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Quantisation of the Kaplan–Yorke model

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Abstract. A classically chaotic model with a strange attractor introduced by Kaplan and Yorke is quantised and analysed in the Heisenberg picture. The equivalence is shown to quantisation in the Schrödinger picture given earlier by one of the authors. As an advantage of the approach presented here, the expectation values of observables can be calculated. It is shown that in expectation values the limits $\hbar \to 0$ and $t \to \infty$ commute.

1. Introduction

The study of quantum systems which are chaotic in their classical limit continues to be of high interest (for a review see Eckhardt (1988)). Chaos in dissipative classical systems is associated with the appearance of strange attractors (Lichtenberg and Lieberman 1983). Classically chaotic quantum models with dissipation can be constructed by the coupling of 'small' quantum systems to heat baths (Graham 1983a, 1985, Graham and Tel 1985, Graham *et al* 1988, Dittrich and Graham 1986). The elimination of the heat bath in some approximations then leads to a master equation for the statistical operator, which has to be solved numerically. Unfortunately, it is not possible, by these methods, to obtain rigorous results because the derivation of the master equation uses perturbation theory with respect to the system–bath interaction and, for classically chaotic systems, the master equation cannot be solved analytically.

It is therefore of interest that an exactly solvable classical model exists, exhibiting a strange attractor (Kaplan and Yorke 1979), which can be quantised exactly, in a special case (Graham 1983a, 1985). In this earlier work the quantisation was given in the Schrödinger picture and directly yielded the master equation for the statistical operator without approximation. The Wigner function of the statistical operator in the steady state could then be constructed exactly and was used to show what happened to the strange attractor in the quantised version. The main reason for solubility was, as usual, an unphysical feature: the existence of a limiting situation of 'global dissipation' where a 'semi-Hamiltonian' treatment (Graham 1985) is possible, with a 'Hamiltonian' which generates time translations in only one 'future' direction (cf section 2).

There are a number of questions which have not yet been answered in the earlier work on this exactly solvable model, which we would like to consider in the present paper. In view of the 'semi-Hamiltonian' nature of the problem it is not clear *a priori* whether a Heisenberg picture can be meaningfully defined. This question seems to be even more important in view of the fact that the results obtained in the Schrödinger picture yielded the Wigner function but could not be used, so far, to obtain explicit

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results for the expectation values of the basic observables whose classical values are readily obtained from the classical model. One might hope, therefore, that quantisation of the model in the Heisenberg picture would be able to yield such results quite directly without the necessity to deal, as an intermediate step, with the complexities of the Wigner distribution corresponding to the classically chaotic state. This expectation is, indeed, borne out by our findings. The new results for the expectation values derived here are used to establish the commutability of the limits $\hbar \to 0$ and $t \to \infty$.

This paper is organised as follows. The next section contains a brief account of the classical model and of some of its properties which we shall need in section 5. Section 3 discusses the quantisation of the model in the Schrödinger picture, adding some new results to those contained in Graham (1983a, 1985). In section 4 the Heisenberg quantisation of the model is derived from the Schrödinger quantisation. Finally, in section 5 we present applications to the evaluation of expectation values and prove the interchangeability of the limits $\hbar \to 0$, $t \to \infty$.

2. The classical model and expectation values

The Kaplan–Yorke model (Kaplan and Yorke 1979) is a discrete dynamical system, described by the family of two-dimensional mappings:

$$x_{n+1} = 2x_n (\mathrm{mod}\ 1) \tag{2.1a}$$

$$y_{n+1} = \lambda y_n + f(x_n) \tag{2.1b}$$

$$x_n \in [0, 1) \qquad y_n \in R \qquad \lambda \in R \qquad n \in N.$$

It is assumed that $|\lambda| < 1$ and that $f: [0, 1) \to R$ be extended to the whole of R by

$$f(x+1) = f(x). \tag{2.2}$$

The Jacobian of the mapping (2.1) being (2λ), the system is dissipative for $|\lambda| < \frac{1}{2}$. If $\lambda = \frac{1}{2}$ there is a local area preservation. In this case, and in order to render the model more interesting, we assume (with Graham (1983a)) that

$$f(x+\tfrac{1}{2}) = f(x). \tag{2.3}$$

Under the above assumption, the mapping (2.1) is non-invertible and we shall refer to this case as 'globally dissipative' (Graham 1983a, 1985). Chaotic behaviour, i.e. sensitive dependence on initial conditions, is guaranteed by the form of (2.1), which allows the computation of the two Lyapunov exponents $\lambda_1 = \ln 2$ and $\lambda_2 = \ln \lambda$. Moreover, it is found (Kaplan and Yorke 1979) that the system exhibits a bounded attractor, which is therefore necessarily 'strange' (Eckmann and Ruelle 1985). The case $\lambda = \frac{1}{2}$ lies at the threshold of the Kaplan and Yorke conjecture concerning the Hausdorff dimension (Kaplan and Yorke 1979), but an extrapolation of the conjecture to this case suggests that the Hausdorff dimension of the attractor equals two if $\lambda = \frac{1}{2}$.

Let (x_0, y_0) be the initial values for the map (2.1). Iterating the map n times, we obtain

$$x_n = 2^n x_0 (\mathrm{mod}\ 1) \tag{2.4a}$$

$$y_n = \lambda^n y_0 + \sum_{l=0}^{n-1} \lambda^l f(2^{n-1-l} x_0). \tag{2.4b}$$

From the above equations the conditional probability $P_n(y|x; y_0)$ may be derived (Graham 1983b):

$$P_n(y|x; y_0) = \frac{1}{2^n}\sum_{m=0}^{2^n-1}\delta\left[y-\lambda^n y_0-\sum_{l=0}^{n-1}\lambda^l f\left(\frac{x+m}{2^{l+1}}\right)\right] \tag{2.5}$$

whence one may conclude that the distribution is, for each n, concentrated on the set of curves:

$$y_m^n(x) = \lambda^n y_0 + \sum_{l=0}^{n-1}\lambda^l f\left(\frac{x+m}{2^{l+1}}\right) \qquad m = 0, 1, \ldots, 2^{n-1}. \tag{2.6}$$

Taking $n\to\infty$ and assuming $|\lambda|<1$, (2.6) leads us to define the attractor as the closure of the set of curves:

$$y_m^\infty = \sum_{l=1}^{\infty}\lambda^{l-1} f\left(\frac{x+m}{2^l}\right) \qquad m = 0, 1, 2, \ldots. \tag{2.7}$$

A mathematically rigorous approach, based on a proper definition of the (ergodic) limit $n\to\infty$ in (2.5), is given in Mayer and Roepstorff (1983): it leads, however, to the same result (2.7).

We shall now compute some classical expectation values, particularly of functions of the variable y—the '(angular) momentum'—which we shall need in section 5. For this purpose we set the initial distribution $P_0(x_0) = 1$ (the uniform distribution) (other choices could be made) and, for definiteness

$$f(x) = -\sin(4\pi x). \tag{2.8}$$

Note that the above f satisfies (2.2) and (2.3). Let p be a positive integer. Then, by (2.4*b*),

$$\langle y_n^p\rangle = \sum_{k=0}^{p}\binom{p}{k}(\lambda^n y_0)^{p-k}\int_0^1 \mathrm{d}x_0\left(-\sum_{l=0}^{n-1}\lambda^l\sin(4\pi 2^{n-1-l}x_0)\right)^k. \tag{2.9}$$

In order to proceed, we need to evaluate the product of k sines. If k is even, it reduces to a sum of cosines, if k is odd it reduces to a sum of sines which yield zero contribution to the integrals. We may thus perform the integration in (2.9), obtaining

$$\langle y_n^p\rangle = \sum_{\substack{k=0\\(k\text{ even})}}^{[p]}\binom{p}{k}(\lambda^n y_0)^{p-k}(-1)^{k/2}(1/2)^k\sum_{l_1=1}^{n-1}\cdots\sum_{l_k=1}^{n-1}\sum_{s_1=-1,1}\cdots\sum_{s_k=-1,1}\lambda^{l_1+\ldots+l_k}s_1\ldots s_k$$
$$\times\,\delta\left(\sum_{q=1}^{k}2^{n-l_q}s_q\right) \tag{2.10}$$

where $[p]$ denotes the greatest even integer smaller or equal to p and δ is the Kronecker delta. Hence

$$\lim_{n\to\infty}\langle y_n^p\rangle \equiv \langle y_\infty^p\rangle = 0 \qquad \text{if } p \text{ is odd.} \tag{2.11}$$

When p is even, for $n\to\infty$ only the term $k=p$ in the first summation survives, and

$$\langle y_\infty^p\rangle = \frac{(-1)^{p/2}}{2^p}\sum_{\{l_i\}}\sum_{\{s_i\}}\lambda^{l_1+\ldots+l_p}s_1\ldots s_p\delta\left(\sum_{q=1}^{p}2^{n-l_q}s_q\right). \tag{2.12}$$

For finite n we have

$$\langle y_n^2 \rangle = \frac{1}{2} \frac{1-\lambda^{2n}}{1-\lambda^2} \tag{2.13}$$

$$\langle y_n^4 \rangle = \lambda^{4n} y_0^4 + 3\lambda^{2n} y_0^2 + \frac{3}{4}\left(\frac{1-\lambda^{2n}}{1-\lambda^2}\right)^2 - \frac{3}{8}\frac{1-\lambda^{4n}}{1-\lambda^4} - \tfrac{3}{2}\lambda^5 \frac{1-\lambda^{(n-2)4}}{1-\lambda^4}. \tag{2.14}$$

The results (2.13) and (2.14) were first obtained by Jensen and Oberman (1981). For the expectation values also involving the variable x, we have

$$\langle x_n^q y_n^p \rangle = \sum_{k=0}^{2^n-1} \int_{k2^{-n}}^{(k+1)2^{-n}} \mathrm{d}x_0 (2^n x_0 - k)^q \left(\lambda^n y_0 - \sum_{l=0}^{n-1} \lambda^l \sin(4\pi 2^{n-1-l} x_0) \right)^p \tag{2.15}$$

if p, q are positive integers. The above expression may be evaluated as before.

3. Schrödinger–Wigner quantisation of the model

Consider, now, the map (2.1) for $\lambda = \frac{1}{2}$, which we rewrite in the form

$$q_{n+1} = 2q_n (\mathrm{mod}\ 1) \tag{3.1a}$$

$$p_{n+1} = \tfrac{1}{2} p_n - g(q_n) \tag{3.1b}$$

where $f = -g'$ satisfies (2.2) and (2.3). This map has been quantised in the Schrödinger picture in Graham (1983a, 1985), to which we refer for details.

Defining the density matrix ρ_n at time n one obtains the master equation (Graham 1983a)

$$\langle q|\rho_{n+1}|q'\rangle = \tfrac{1}{2} \exp\left(-\frac{2\mathrm{i}}{\hbar}(g(q/2) - g(q'/2))\right)\left(\left\langle \frac{q}{2}\middle|\rho_n\middle|\frac{q'}{2}\right\rangle + \left\langle \frac{q+1}{2}\middle|\rho_n\middle|\frac{q'+1}{2}\right\rangle\right). \tag{3.2}$$

Introducing the Wigner function

$$W_n(p, q) = \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{2\pi\hbar} \exp\left(-\frac{\mathrm{i}px}{\hbar}\right) \langle q + x/2|\rho_n|q - x/2\rangle \tag{3.3}$$

one obtains (Graham 1983a)

$$\begin{aligned} W_\infty(p, q) = \lim_{n\to\infty} 2^{-n} \sum_{m=0}^{2^n-1} \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{2\pi} \exp(-\mathrm{i}px) \\ \times \exp\left\{-\frac{2\mathrm{i}}{\hbar} \sum_{l=1}^{n} \left[g\left(\frac{(q+\frac{1}{2}\hbar x)\ (\mathrm{mod}\ 1) + m}{2^l}\right)\right.\right. \\ \left.\left. - g\left(\frac{(q-\frac{1}{2}\hbar x)(\mathrm{mod}\ 1) + m}{2^l}\right)\right]\right\}. \end{aligned} \tag{3.4}$$

The classical limit of the asymptotic Wigner function is obtained upon performing

$$\lim_{\hbar\to 0} W_\infty(p, q) = \lim_{\hbar\to 0} \lim_{n\to\infty} W_n(p, q).$$

If we formally interchange these limits in (3.4), we obtain, upon expanding g up to the first order in $\hbar$ and doing the integral in x,

$$\lim_{\hbar\to 0} W_\infty(p,q) = \lim_{n\to\infty} 2^{-n} \sum_{m=0}^{2^n-1} \delta(p - F_m(q)) \tag{3.5}$$

$$F_m(q) = -\sum_{l=0}^{\infty} 2^{-l} g\left(\frac{q+m}{2^{l+1}}\right). \tag{3.6}$$

Comparing with (2.5) we see that the correct classical result obtains. However, it is difficult to make rigorous this interchange of the limits $\hbar \to 0$, $n \to \infty$. Therefore, a different strategy based on the Heisenberg picture is followed in section 4, which demonstrates the interchangeability of the two limits at least for all moments.

The next term in the expansion of g is third order in $\hbar$, because the second-order terms cancel:

$$W_{\mathrm{SC}\infty}(p,q) = \lim_{n\to\infty} 2^{-n} \sum_{m=0}^{2^n-1} (\Delta_m(q))^{-1/3} \,\mathrm{Ai}\left(\frac{p - F_m(q)}{(\Delta_m(q))^{1/3}}\right) \tag{3.7}$$

where SC stands for 'semiclassical', Ai for the Airy function, and

$$\Delta_m(q) = \hbar^2 \sum_{k=1}^{\infty} 2^{-3k-2} g'''\left(\frac{q+m}{2^k}\right). \tag{3.8}$$

The above result shows that each branch $p = F_m(q)$ of the classical attractor is delocalised with a width of order $(\Delta_m(q))^{1/3}$, and hence proportional to $\hbar^{2/3}$. For this reason, the attractor does not survive quantisation (see Graham (1983a) for a full discussion).

One question one is tempted to ask in the general framework of Schrödinger–Wigner quantisation, as applied to the present model, is in what respects the semiclassical behaviour of the Wigner function differs from its well studied counterpart for integrable systems (Berry 1977, Ozorio de Almeida 1988). For one-dimensional integrable systems just three types of semiclassical behaviour are found, according to whether two, three or four stationary points (corresponding to a stationary phase analysis of the integral) coalesce ('fold catastrophe', 'cusp catastrophe' and 'swallow-tail catastrophe', respectively—see Berry (1977)). The semiclassical behaviour corresponding to the 'fold catastrophe' shows up in (3.7) and (3.8): (3.7) follows from (3.4) upon expanding g up to third order in $\hbar$, and performing a change of variable in the x integration. The latter results in the appearance of the Airy function, but is allowed only if $\Delta_m(q) \neq 0$, where Δ_m is defined by (3.8). If $\Delta_m(q) = 0$ for some $q \in [0, 1)$, and some fixed m, we expand g up to fifth order in $\hbar$ (the fourth-order terms cancel) and perform the limit $n \to \infty$ formally under the integral in (3.4). Define

$$\Gamma_m(q) \equiv \sum_{l=1}^{\infty} \frac{1}{2^{5l+3}} g^{(5)}\left(\frac{q+m}{2^l}\right) \tag{3.9}$$

where $g^{(5)}$ denotes the fifth-order derivative of g, and suppose that, for fixed $m = m_0$ and q, $\Gamma_{m_0}(q) \neq 0$. Then the leading $\hbar$ contribution of this term to (3.4) may be written as

$$W_{\mathrm{SC}}^{m_0}(p,q) = \frac{1}{2^n} \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{2\pi} \exp\left[-\mathrm{i}\left(x(p + F_{m_0}(q)) + \frac{\hbar^4 x^5}{5!}\Gamma_{m_0}(q)\right)\right] \tag{3.10}$$

where F_m is given by (3.7).

Note that both $\Delta_{m_0}(\ )$ and $\Gamma_{m_0}(\ )$ are almost-periodic functions (Bohr 1947). It is expected that they possess no common zeros, but for the moment we need only pick some q which is a zero of Δ_{m_0} but not of Γ_{m_0}: we have checked numerically that such q exist for $m_0=0$. Performing the change of variable $x' = \hbar^{+4/5}x(\Gamma_m(q))^{1/5}$ upon (3.10) we obtain

$$W^{m_0}_{\mathrm{SC}}(p,q)=\frac{1}{2^n}\frac{\hbar^{-4/5}}{(\Gamma_m(q))^{1/5}}\int_{-\infty}^{\infty}\frac{\mathrm{d}x'}{2\pi}\exp\left[-\mathrm{i}\left(\frac{x'(p-F_{m_0}(q))}{(\Gamma_{m_0}(q))^{1/5}\hbar^{4/5}}+\frac{x'^5}{5!}\right)\right]. \tag{3.11}$$

Hence if the function

$$h(\lambda)\equiv\int_{-\infty}^{\infty}\mathrm{d}x\,\exp\left[-\mathrm{i}\left(x\lambda+\frac{1}{5!}x^5\right)\right]$$

is bounded (which may be checked by contour integration techniques similar to those employed to prove the same assertion for the Airy function) we see that $W^{m_0}_{\mathrm{SC}}$ in (3.11) attains a maximum value of order $\hbar^{-4/5}$, typical of the 'swallow-tail catastrophe' (Berry 1977). Nevertheless, in order that this behaviour, derived above for fixed $m=m_0$ (the corresponding curve being the analogue of the torus in the integrable systems treated by Berry), manifested itself globally in the Wigner function (3.4), it would be necessary that the almost-periodic function

$$\Delta(\zeta)=\sum_{k=1}^{\infty}2^{-3k-2}g'''\left(\frac{\zeta}{2^k}\right)$$

had zeros in a subset of $\{\zeta_0, \zeta_0+1, \ldots, \zeta_0+2^n-1\}$, whose number of points increased proportionally to 2^n, as $n\to\infty$, where ζ_0 is a zero of $\Delta(\ )$. This fact is due to the average $\lim_{n\to\infty}2^{-n}\sum_{m=0}^{2^n-1}$ in (3.4). Due to almost-periodicity of $\Delta(\ )$ as opposed to the periodicity found in the analogous integrable systems, this possibility is excluded. Hence the swallow-tail catastrophe, which is generic for one-dimensional integrable systems in a cylindrical space (Berry 1977), does not show up in this model. Other types of behaviour are excluded by the non-existence of common zeros of Δ_m and Γ_m—a fact which may be safely assumed to be true, but is difficult to prove, and cannot even be verified numerically because numerical analysis enforces truncations which render the various almost-periodic functions periodic. Summarising, the main qualitative difference between the present model and integrable systems lies in the 'average over toruses' and the appearance of almost-periodic functions in (3.4).

As a last remark, note that the Wigner function defined in Berry (1977) as appropriate to a cylindrical space is given in the present model by

$$\hat{W}_n(p_l,q)=\int_{-1/2}^{1/2}\mathrm{d}x\,\exp(-2\pi\mathrm{i}lx)\langle q+x/2|\rho_n|q-x/2\rangle \tag{3.12}$$

where $p_l=2\pi\hbar l$, where l is integer. The two definitions (3.3) and (3.12) are two alternative representations of the statistical operator ρ_n. Inverting (3.3) and (3.12) we obtain

$$\langle q+x/2|\rho_n|q-x/2\rangle=\int_{-\infty}^{+\infty}\mathrm{d}p\,\exp\left(\mathrm{i}\frac{px}{\hbar}\right)W_n(p,q) \tag{3.13}$$

$$\left\langle q+\frac{x(\mathrm{mod}\ 1)}{2}|\rho_n|q-\frac{x(\mathrm{mod}\ 1)}{2}\right\rangle=\sum_{l=-\infty}^{+\infty}\exp(2\pi\mathrm{i}lx)\,\hat{W}_n(p_l,q) \tag{3.14}$$

respectively. Using (3.13) on the right-hand side of (3.12) we may express $\hat{W}_n$ in terms of W_n as

$$\hat{W}_n(p_l, q) = 2\hbar \int_{-\infty}^{+\infty} \mathrm{d}p \frac{(-1)^l \sin(p/2\hbar)}{(p - 2\pi l\hbar)} W_n(p, q). \tag{3.15}$$

As the left-hand side of (3.13) is periodic in x with period 2 only (*not* period 1) it follows from (3.3) that $W_n(p, q)$ may be written as

$$W_n(p, q) = \sum_{k=-\infty}^{+\infty} P_n(k, q)\delta(p - \pi\hbar k) \tag{3.16}$$

where

$$P_n(k, q) = \int_0^1 \mathrm{d}x \langle q + x|\rho_n|q - x\rangle \exp(-2\pi \mathrm{i} kx). \tag{3.17}$$

With $|q+1\rangle = |q\rangle$, (3.17) implies

$$P_n(k, q + \tfrac{1}{2}) = (-1)^k P_n(k, q). \tag{3.18}$$

Using (3.16) in (3.15) we find

$$\hat{W}_n(p_l, q) = P_n(2l, q) + \sum_{m=-\infty}^{+\infty} \frac{(-1)^{m+1}}{(m - l + \frac{1}{2})\pi} P_n(2m+1, q). \tag{3.19}$$

Vice versa, we may use (3.14) on the right-hand side of (3.3) to express W_n in terms of $\hat{W}_n$. However, before inserting (3.14) it is necessary to re-express (3.3) as

$$\begin{aligned} W_n(p, q) = \int_{-1/2}^{1/2} \frac{\mathrm{d}x}{2\pi\hbar} \sum_{m=-\infty}^{+\infty} \left(\exp - \frac{\mathrm{i}px}{\hbar} - \frac{\mathrm{i}2mp}{\hbar}\right)\bigg[\left\langle q + \frac{x(\mathrm{mod}\ 1)}{2}\middle|\rho_n\middle| q - \frac{x(\mathrm{mod}\ 1)}{2}\right\rangle \\ + \exp\left(\frac{-\mathrm{i}p}{\hbar}\right)\left\langle q + \tfrac{1}{2} + \frac{x(\mathrm{mod}\ 1)}{2}\middle|\rho_n\middle| q + \tfrac{1}{2} - \frac{x(\mathrm{mod}\ 1)}{2}\right\rangle\bigg] \end{aligned} \tag{3.20}$$

where the summation over the integer m is introduced to compensate the restriction of the integration over x to the interval $[-\frac{1}{2}, \frac{1}{2}]$. The summation over m may be carried out by the Poisson summation formula and we obtain a result of the form (3.16) with

$$\begin{aligned} &P_n(k, q) = \tfrac{1}{2}(\hat{W}_n(p_k/2, q) + \hat{W}_n(p_k/2, q + \tfrac{1}{2})) && k \text{ even} \\ &P_n(k, q) = \tfrac{1}{2} \sum_{l=-\infty}^{+\infty} \frac{(-1)^{l+(k-1)/2}}{\pi(k/2 - l)} (\hat{W}_n(p_l, q) - \hat{W}_n(p_l, q + \tfrac{1}{2})) && k \text{ odd.} \end{aligned} \tag{3.21}$$

Thus the two Wigner functions W and $\hat{W}$ are both valid representations of the statistical operators. In order to calculate expectation values of observables with the help of the Wigner function the observables have to be Weyl ordered but, obviously, the rules for Weyl ordering differ in the two cases. The Weyl order of an observable A corresponding to (3.3) is

$$A(p, q) = \int_{-\infty}^{+\infty} \frac{\mathrm{d}x}{2\pi\hbar} \exp(-\mathrm{i}px)\langle q + x/2|A|q - x/2\rangle \tag{3.22}$$

while that corresponding to (3.12) is

$$\hat{A}(p_l, q) = \int_{-1/2}^{1/2} \mathrm{d}x \exp(-\mathrm{i}p_l x)\langle q + x/2|A|q - x/2\rangle. \tag{3.23}$$

The two representations $A(p, q)$, $\hat{A}(p_l, q)$ may again be expressed in terms of each other. The expectation value of A is given by the two equivalent expressions:

$$\langle A_n \rangle = \int_{-\infty}^{+\infty} \mathrm{d}p \int_0^1 \mathrm{d}q \, A(p, q) W_n(p, q) = \sum_{l=-\infty}^{\infty} \int_0^1 \mathrm{d}q \, \hat{A}(p_l, q) \hat{W}_n(p_l, q) \tag{3.24}$$

as may be verified by using the relations given above. The two alternative representations differ only by terms which, in the semiclassical limit, oscillate infinitely rapidly with respect to the angular momentum, as can be seen explicitly in (3.18) and (3.21). In the asymptotic expansion (3.7) such terms do not contribute. Therefore, asymptotically, $W(p, q)$ and $\hat{W}(p, q)$ are both represented by (3.7).

As we saw in this section, for the model under investigation, the Wigner phase-space density in the semiclassical limit is as powerful as in the case of integrable systems, allowing for results of comparable generality. A mathematically rigorous proof of the expected interchangeability of the limits $\hbar \to 0$ in $n \to \infty$ using the Wigner function is, however, not easy, as remarked after (3.4), and the actual computation of expectation values of observables is rather awkward in the Wigner formulation. These disadvantages are, however, exactly the advantages of the Heisenberg picture which we now consider.

4. Heisenberg picture

The equation of motion (3.2) in the Schrödinger picture is written in the eigenrepresentation of the phase operator $\hat{q}$ which, following Judge and Lewis (1963) (cf also Carruthers and Nieto (1968)), we define by

$$\hat{q}|q\rangle = q(\mathrm{mod}\ 1)|q\rangle \tag{4.1}$$

with

$$q(\mathrm{mod}\ 1) = q - \sum_{m=1}^{\infty} \Theta(q-m) + \sum_{m=0}^{\infty} \Theta(-q+m).$$

We note the orthogonality relation:

$$\langle q|q'\rangle = \sum_{m=-\infty}^{+\infty} \delta(q-q'-m) \equiv \delta^{(1)}(q-q'). \tag{4.2}$$

For later use we also note the properties, for integer m,

$$\delta^{(1)}(x+m) = \delta^{(1)}(x) \qquad \delta^{(1)}(mx) = 1/m \sum_{k=0}^{m-1} \delta^{(1)}(x-k/m). \tag{4.3}$$

Furthermore, we give the commutation relation of the angular momentum operator $\hat{p} = \mathrm{i}\hbar\, \mathrm{d}/\mathrm{d}q$ and the phase operator $\hat{q}$

$$[\hat{p}, \hat{q}] = \hbar/\mathrm{i} - \hbar/\mathrm{i}\, \delta^{(1)}(q). \tag{4.4}$$

In order to define a Heisenberg picture we consider the evolution of the expectation value of an arbitrary observable Ω. The matrix elements of Ω, $\langle q|\Omega|q'\rangle$, are 1-periodic in q and in q'.

From

$$\langle \Omega_n \rangle = \mathrm{Tr}\, \rho_n \Omega \tag{4.5}$$

we obtain

$$\langle\Omega_{n+1}\rangle=\int_0^1 \mathrm{d}q\int_0^1 \mathrm{d}q'\langle q|\Omega|q'\rangle\langle q'|\rho_{n+1}|q\rangle \tag{4.6}$$

which we may use to define $\langle q|\Omega_{n+1}|q'\rangle$ via

$$\langle\Omega_{n+1}\rangle=\int_0^1 \mathrm{d}q\int_0^1 \mathrm{d}q'\langle q|\Omega_{n+1}|q'\rangle\langle q'|\rho|q\rangle \tag{4.7}$$

where ρ is the statistical operator at time $n=0$. Using (3.2) in (4.6) and using (4.7) and the fact that $\langle q'|\rho|q\rangle$ is an arbitrary initial state we obtain

$$\langle q|\Omega_{n+1}|q'\rangle=2\exp\left(\frac{2\mathrm{i}}{\hbar}(g(q)-g(q'))\right)[\Theta(\tfrac{1}{2}-q)\Theta(\tfrac{1}{2}-q')+\Theta(q-\tfrac{1}{2})\Theta(q'-\tfrac{1}{2})]\langle eq|\Omega_n|2q'\rangle. \tag{4.8}$$

Equation (4.8) is the equation of motion of an arbitrary observable in the q representation and, together with (4.7), defines the Heisenberg picture. We may now evaluate (4.8) for the operators $\Omega=\hat{q}$ and $\Omega=\hat{p}$, respectively. The calculation is greatly simplified by considering (4.6) in the $\hat{q}_n$ representation. Then we find

$$\begin{aligned}\langle q_n|\hat{q}_{n+1}|q_n'\rangle=&(2q_n')(\mathrm{mod}\ 1)[\delta^{(1)}(q_n-q_n')+\delta^{(1)}(q_n-q_n'-\tfrac{1}{2})]\\&\times[\Theta(\tfrac{1}{2}-q_n)\Theta(\tfrac{1}{2}-q_n')+\Theta(q_n-\tfrac{1}{2})\Theta(q_n'-\tfrac{1}{2})]\end{aligned} \tag{4.9}$$

where we have used (4.3) to simplify $\delta^{(1)}(2x)$. Due to the presence of the Θ functions, the function $\delta^{(1)}(q_1-q_2-\frac{1}{2})$ in (4.9) does not contribute and can therefore be dropped. Then, the values of q and q' in the Θ functions can be taken as equal to each other and hence the Θ functions containing q' can be replaced by 1. After that, the two remaining Θ functions containing q simply add up to 1. We obtain

$$\langle q_n|\hat{q}_{n+1}|q_n'\rangle=(2q_n')(\mathrm{mod}\ 1)\delta^{(1)}(q_n-q_n'). \tag{4.10}$$

Therefore $\hat{q}_{n+1}$ is diagonal in the $\hat{q}_n$ representation and its eigenvalues in that representation satisfy

$$q_{n+1}=(2q_n)(\mathrm{mod}\ 1). \tag{4.11}$$

However, due to the modulo presciption on the right-hand side a simple operator equation independent of the q representation cannot be given. Let us now choose $\Omega_n=\hat{p}_n$ in (4.6). We then obtain for $q_n, q_n'\in[0,\frac{1}{2})$ and $q_n, q_n'\in[\frac{1}{2},1)$

$$\langle q_n|\hat{p}_{n+1}|q_n'\rangle=\tfrac{1}{2}\exp\left(\frac{2\mathrm{i}}{\hbar}(g(q_n)-g(q_n'))\right)\frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial q_n}[\delta^{(1)}(q_n-q_n')] \tag{4.12}$$

and for $q_n\in[0,\frac{1}{2})$, $q_n'\in[\frac{1}{2},1)$ or $q_n\in[\frac{1}{2},1)$, $q_n'\in[0,\frac{1}{2})$ we have

$$\langle q_n|\hat{p}_{n+1}|q_n'\rangle=0. \tag{4.13}$$

Equations (4.12) and (4.13) may be combined to yield

$$\langle q_n|\hat{p}_{n+1}|q_n'\rangle=\left(\frac{1}{2}\frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial q_n}-g'(q_n)\right)\delta^{(1)}(q_n-q_n') \tag{4.14}$$

which can be written in operator form independent of the representation used as

$$\hat{p}_{n+1}=\tfrac{1}{2}\hat{p}_n-g'(\hat{q}_n). \tag{4.15}$$

Equations (4.10) and (4.15) are the equations of motion of the Kaplan-Yorke model in the Heisenberg picture. As in the case of the Schrödinger picture the parameter λ of the classical model (2.1) must be restricted to $\lambda=\frac{1}{2}$. The case $\lambda\neq\frac{1}{2}$ can no longer be treated exactly and has been analysed, in the Schrödinger picture, in Graham (1985).

Equations (4.10) and (4.15) are easily solved. It follows from (4.10) that $\hat{q}_n$ is diagonal in the q_0 representation and acts, in that representation, as the multiplication operator

$$\hat{q}_n=(2^n q_0)(\mathrm{mod}\ 1). \tag{4.16}$$

Then it follows from (4.15) in the same representation that

$$\hat{p}_n=(1/2)^n\hat{p}_0-\sum_{l=0}^{n-1}(1/2)^n g'(2^{n-l-1}q_0) \tag{4.17}$$

with

$$\hat{p}_0=-\mathrm{i}\hbar\ \mathrm{d}/\mathrm{d}q_0. \tag{4.18}$$

These expressions are used, in the following section, for evaluation of expectation values.

5. Expectation values

We now present results on expectation values and establish the interchangeability of the limits $n\to\infty$, $\hbar\to 0$. For the sake of concreteness we shall always assume that the initial state of the system at $n=0$ is the state of vanishing angular momentum

$$\hat{p}_0|0\rangle=0 \qquad \langle q_0|0\rangle=1. \tag{5.1}$$

Furthermore, we specialise to the case

$$g'(x)=\sin(4\pi x) \tag{5.2}$$

but any infinitely differentiable function with the required periodicity property would be allowed in our proof. First we consider expectation values of powers of $\hat{p}_n$. It is convenient to define the function

$$S_n(q_0)=\sum_{l=0}^{n-1}(1/2)^l\sin(4\pi 2^{n-l-1}q_0). \tag{5.3}$$

We note the symmetry

$$S_n(q_0)=-S_n(1-q_0). \tag{5.4}$$

Then we may write

$$\langle 0|\hat{p}_0^r|0\rangle=\int_0^1 \mathrm{d}q_0\left(-\frac{\mathrm{i}\hbar}{2^n}\frac{\mathrm{d}}{\mathrm{d}q_0}-S_n(q_0)\right)^r 1. \tag{5.5}$$

For r odd the integrand changes sign under transforming $q_0\to 1-q_0$ and the integral therefore vanishes, in agreement with the corresponding classical result. We now consider the case of even r. Starting with $r=2$ we have

$$\langle 0|\hat{p}_n^2|0\rangle=\langle 0|\left(\frac{\hat{p}}{2^n}-S_n(q_0)\right)\left(\frac{\hat{p}_0}{2^n}-S_n(q_0)\right)|0\rangle. \tag{5.6}$$

Using the properties $\hat{p}_0|0\rangle = 0$, $\langle 0|\hat{p}_0 = 0$ we obtain

$$\langle 0|\hat{p}_n^2|0\rangle = \int_0^1 \mathrm{d}q_0\, S_n^2(q_0) \tag{5.7}$$

which is the classical result (2.9) and (2.13). Turning to $r=4$ and biinging the $\hat{p}_0$ factors symmetrically to the left and right, we find

$$\langle 0|\hat{p}_n^4|0\rangle = \langle 0|S_n^4(q_0)|0\rangle - \frac{1}{2^{2n}}\langle 0|([\hat{p}_0, S_n(q_0)])^2|0\rangle \tag{5.8}$$

where

$$[\hat{p}_0, S_n(q_0)] = -\mathrm{i}\hbar S_n'(q_0) \tag{5.9}$$

and we have used the fact that $[\hat{p}_0, S_n(q_0)]$ commutes with functions of q_0. The first term on the right-hand side of (5.8) is just the classical result, as follows from (2.9). The second term is a quantum correction which is proportional to $\hbar^2$. By methods analogous to those used for the classical expectation values this quantum correction may be evaluated explicitly and we obtain

$$\langle p_n^4\rangle = \langle y_n^4\rangle + 2\hbar^2\pi^2\frac{1-(1/2)^{4n}}{1-(1/2)^4}. \tag{5.10}$$

The same method may be applied to the calculation of moments of $\hat{p}_n$ of higher order. The result is of the form (for r even)

$$\langle \hat{p}_n^r\rangle = \langle y_n^r\rangle + \sum_{m=2}^{r/2} \hbar^{2(m-1)} C_m(n). \tag{5.11}$$

The coefficients $C_m(n)$ are finite and bounded for all n. To see this, we first note that, for finite n, $C_m(n)$ is calculable as the average of a finite product of functions $S_n(q_0)$ and derivatives of $S_n(q_0)$ of up to mth order. The kth derivative $S_n^{(k)}(q_0)$ of $S_n(q_0)$ (with $k \leqslant m$) is bounded by

$$|S_n^{(k)}(q)| \leqslant \sum_{l=0}^{n-1} (2\pi)^k 2^{kn-kl} = (2\pi)^k \frac{2^{kn}-1}{1-(1/2)^k}. \tag{5.12}$$

In the expression for $C_m(n)$ the derivative $S_n^{(k)}(q)$ appears multiplied by an additional factor 2^{-kn} because, due to (4.17), each $\hat{p}_0$ appears multiplied by 2^{-n}. Therefore, $C_m(\infty) = \lim_{n\to\infty} C_m(n)$ will exist if $\lim_{n\to\infty} 2^{-kn} S_n^{(k)}(q)$ exists, which is guaranteed by the bound (5.12) on $|S_n^{(k)}(q)|$. We may therefore perform the limit $n\to\infty$ in (5.11) *before* taking $\hbar \to 0$ and obtain

$$\langle \hat{p}_\infty^r\rangle = \langle y_\infty^r\rangle + \sum_{m=2}^{r/2} \hbar^{2(m-1)} C_m(\infty). \tag{5.13}$$

Taking now $\hbar \to \infty$ we obtain the classical result, which shows that the two limits commute. This result is trivially extended to arbitrary moments of $\hat{q}_n$, which receive no quantum corrections at all because (4.11) is uncoupled from $\hat{p}$ and identical to the classical equation (2.1*a*). Mixed moments of $\hat{p}_n$ and $\hat{q}_n$ may again be handled by commutator methods. Again it is easy to show that the quantum corrections remain finite for $n\to\infty$ and vanish by taking $\hbar\to 0$ afterwards. The convergence of all stationary moments ($n\to\infty$) to the classical moments implies that the Wigner phase-space density evaluated for $n \to \infty$ must converge to the classical phase-space probability density 'in distribution' (equality of all moments).

Rigorous results on the semiclassical limit of *Hamiltonian* systems (Hepp 1974, Ginibre and Velo 1980, Blanchard and Sirigue 1985) show that the limits $\hbar \to 0$ and $t \to \infty$ (in our case $n \to \infty$) never commute. Our result is not in conflict with these rigorous results because of the non-Hamiltonian nature of the dynamics of the Kaplan–Yorke model even for $\lambda = \frac{1}{2}$. In the Schrödinger picture the non-Hamiltonian nature appeared by the necessity to consider mixtures rather than pure states and to formulate the dynamics by a master equation. In the Heisenberg picture the equations of motion for $\hat{q}_n$ and $\hat{p}_n$, even though conserving the commutation relations, can also not be derived from a Hamiltonian. The most important consequence is the lack of invertibility of the quantised dynamics, which means that loss of information is inextricably tied to the time evolution in this model. This latter aspect of our quantum model is shared by all dissipative quantum systems, i.e. our results which are based on this property can be expected to be typical also for physically more realistic dissipative systems, which are chaotic in their classical limit. It is the loss of information and the connected increase in entropy which drives our system to a unique steady state whose classical limit conicides with the classical steady state. More relatistic quantum systems with dissipation are modelled by coupling to reservoirs and tracing the density matrix over the reservoirs. Then a loss of information is again connected with the time evolution, and one may expect that the limits $\hbar \to 0$ and $t \to \infty$ commute even though, in cases involving chaotic classical dynamics, a rigorous proof will be much more difficult than in the simple model considered here.

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